A Note on the Eigenvalue Density of Random Matrices

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ABSTRACT

The distribution of eigenvalues of $N \times N$ random matrices in the limit $N \to \infty$ is the solution to a variational principle that determines the ground state energy of a confined fluid of classical unit charges. This fact is a consequence of a more general theorem, proven here, in the statistical mechanics of unstable interactions. Our result establishes the eigenvalue density of some ensembles of random matrices which were not covered by previous theorems.

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I. INTRODUCTION

Since the pioneering work of Wigner [37,38,39], there has been a considerable effort to understand the statistics of eigenvalues of $N \times N$ random matrices. The problem has three scales: (i) the density of eigenvalues which converges to a deterministic limit as $N \to \infty$; (ii) the fluctuations of order one around this deterministic density; (iii) the local statistics on the scale of the typical distance between eigenvalues. Item (i) depends on the particular matrix ensemble while (ii) and (iii) are "universal," in the sense that they depend only on some overall matrix characteristics (e.g. the matrices being real and symmetric). The "classical" results are reviewed in [28]; see also [31] for a collection of early work. Recently, in the context of the double scaling limit of 2D quantum gravity [10], (ii) and (iii) have been studied at the edge of the support of the density of states where novel universality classes occur. Among recent work on (ii) we also mention [29,34], and [30,4,9] regarding (iii). In our paper we will consider only the largest scale (i).

For various $N \times N$ random matrix ensembles of the form

$$m^{(N)}(dM) = \mathbf{Q}(N)^{-1} e^{-\kappa N \operatorname{Tr} V(M)} dM\,, \tag{1.1}$$

the joint probability distribution for the N eigenvalues $\lambda_1, ..., \lambda_N$ (which may be real or complex) is identical to the (configurational) canonical ensemble at inverse temperature β of N unit point charges at positions $\lambda_1, ..., \lambda_N \in \Lambda \subset \mathbb{R}^2$. The region Λ can be all of $\mathbb{R}^2$, the unit disk $\mathrm{B}_1 \subset \mathbb{R}^2$, the unit circle $\mathbb{S}^1$, the entire real line $\mathbb{R}$, or some other set, depending on the type of random matrices. This joint probability distribution has the general form

$$d\mu^{(N)} = Q^{(N)}(\beta)^{-1} \exp\bigl(-\beta H^{(N)}\bigr) d\lambda_1 \cdots d\lambda_N \tag{1.2}$$

on Λ^N, where $d\lambda_k$ is the uniform measure on Λ and $Q^{(N)}(\beta)$ the normalizing partition function. The classical Hamiltonian, $H^{(N)}$, is of the form

$$H^{(N)}(\lambda_1, ..., \lambda_N) = \sum_{1 \le j < k \le N} G(\lambda_j, \lambda_k) + \sum_{1 \le k \le N} F(\lambda_k) + NV(\lambda_k)\,, \tag{1.3}$$

where $G(\lambda_j, \lambda_k) = G(\lambda_k, \lambda_j)$ is $(2\pi\times)$ a Green's function for $-\Delta$ in two dimensions, $F(\lambda) = \lim_{\eta \to \lambda} (G(\lambda, \eta) + \ln|\lambda - \eta|)$ is the regular part of G, and $V(\lambda)$ given in (1.1).

Let us list a few examples. The joint eigenvalue distribution, $\mu^{(N)}$, of the Gaussian ensembles (i.e., $V(M) = M^\dagger M$ in (1.1)) for the real symmetric [22], the complex Hermitian [14], the general complex [20], and the Hermitian self-dual quaternionic [14] $N \times N$ random matrices takes the form

$$d\mu^{(N)} = \frac{1}{Q^{(N)}(\beta)} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta \prod_{1\le k\le N} e^{-\beta N|\lambda_k|^2} d\lambda_k\,; \tag{1.4}$$

see also [28]. Clearly, this corresponds to (1.2), (1.3), where $G(\lambda_1, \lambda_2) = -\ln|\lambda_1 - \lambda_2|$ is the free space Green's function (whence $F \equiv 0$) and $V(\lambda) = |\lambda|^2$ a quadratic potential. The eigenvalues for real symmetric, complex Hermitian, and Hermitian self-dual quaternionic matrices are real. Therefore, the charges are confined to $\mathbb{R}$, i.e. $\Lambda = \mathbb{R}$. The parameters have the values $\beta = 1, 2, 4$ and $\kappa = 1, 2, 2$, respectively. For each of the associated ensembles of unitary matrices the charges are confined to the unit circle, i.e. $\Lambda = \mathbb{S}^1$. Then β is unmodified but $\kappa = 0$ [13]. For general complex random matrices, the eigenvalues are complex, corresponding to unconfined charges, i.e. $\Lambda = \mathbb{R}^2 = \mathbb{C}$, and $\beta = 2$, $\kappa = 2$ [20]. The joint eigenvalue distribution for the general real quaternionic $N \times N$ matrices takes the more complicated form [20]

$$d\mu^{(N)} = \frac{1}{Q^{(N)}(\beta)} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta |\lambda_i - \lambda_j^*|^\beta \prod_{1\le k\le N} |\lambda_k - \lambda_k^*|^\beta e^{-\beta N|\lambda_k|^2} d\lambda_k \tag{1.5}$$

with $\beta = 2$, $\kappa = 1$, $\Lambda = \mathbb{R}^2 = \mathbb{C}$. Although apparently not noticed previously, (1.5) can be interpreted as a configurational canonical Coulomb ensemble, with a Hamiltonian (1.3) in which now $G(\lambda_1, \lambda_2) = -\ln|\lambda_1 - \lambda_2| - \ln|\lambda_1 - \lambda_2^*|$ is the Green's function of $-\Delta$ for the upper half space $\mathbb{R} \times \mathbb{R}^+ = \mathbb{C}^+$ equipped with a perfectly dia-electric condition at its boundary $\partial\mathbb{C}^+$ (the real axis) and asymptotic free conditions at infinity, extended symmetrically to $\mathbb{R}^2$, having a regular part given by $F(\lambda) = -\ln(2|\mathcal{I}m(\lambda)|)$. Moreover, $V(\lambda) = |\lambda|^2$. This Hamiltonian describes N Coulomb point charges interacting via the free space Green's function $-\ln|\lambda - \lambda'|$ amongst each other and also with N identical image charges with respect to the line $\mathcal{I}m(\lambda) = 0$. Since the interaction of a charge with its own image contributes only an amount $F/2$ to the Hamiltonian, $F/2 + NV$ is now to be counted as the external potential.

For symmetric random matrices, properties of the eigenvalue statistics have been computed in great detail for arbitrary N, using explicit expansion techniques [28, 27, 20], group theoretical methods [13], the method of orthogonal polynomials [28,30,4,9], as well as some more recent developments in soliton theory and two-dimensional quantum gravity [1,2]. The beautiful connection to two-dimensional Coulomb systems suggests to use the general methods of statistical mechanics when the above algebraic methods fail. Even in the exactly solvable situations the statistical mechanics approach may provide us rather readily with certain relevant asymptotic ($N \to \infty$) results which to extract from the exact finite N solutions would require quite tedious and lengthy computations. Furthermore, as Dyson has pointed out [13], in the framework of statistical mechanics the limit $N \to \infty$ makes sense for arbitrary β and not only for the discrete values of $\beta = 1, 2, 4$. Thereby one achieves a "thermodynamic view point" which yields valuable new insights into random matrices. Early results in this direction are in [13,40], and more recent ones in [8,17,18].

The prime example of the exploitation of the Coulomb analogy is Wigner's [39] electrostatic derivation of his semi-circle law

$$\rho(\lambda) = \begin{cases} (2/\pi)(1-\lambda^2)^{1/2}\,; & |\lambda| < 1 \\ \qquad 0 \qquad ; & |\lambda| \geq 1 \end{cases} \tag{1.6}$$

for the eigenvalue density $\rho(\lambda)$ in the limit $N = \infty$ of the Gaussian ensemble (1.1) in the case of real symmetric random matrices. Wigner [39] argued, heuristically, that when $N \to \infty$, the eigenvalue density for (1.4) can be obtained from a variational principle for a continuum charge density $\rho(\lambda)$ of total charge 1, restricted to the real line, that satisfies the requirement of mechanical force balance between its own electrostatic force field and the applied force field $-\partial_\lambda |\lambda|^2 = -2\lambda$. Previously [37] he had proved, by the method of moments, that (1.6) holds true for a Bernoulli ensemble of bordered random sign matrices, which suggested that (1.6) was the limiting law under more general circumstances [39]. In [38,39] he announced that (1.6) can indeed be proved to hold for a wider class of ensembles under a mild set of conditions, including the Gaussian real symmetric ensembles.

Interestingly in itself, entropy plays no role in Wigner's variational principle, which is concerned only with the ground state energy of the classical continuum Coulomb fluid. To get an intuitive idea how this can arise from the canonical measure (1.4) with *fixed* β

($< \infty$), we rewrite (1.4) as

$$d\mu^{(N)} = \frac{1}{Q^{(N)}(\beta)} \exp\left(\beta N \left[\frac{1}{N} \sum_{1 \le i < j \le N} \ln |\lambda_i - \lambda_j| - \sum_{k=1}^{N} |\lambda_k|^2 \right] \right) \prod_{1 \le k \le N} d\lambda_k \,. \tag{1.7}$$

Apparently, the limit $N \to \infty$ for (1.7) is a *simultaneous* thermodynamic *and* zero-temperature limit for an unstable Hamiltonian with mean-field scaling. If in (1.7) we replace βN by βN_0, with N_0 fixed, and then let $N \to \infty$, we obtain the variational principle of [6,23] for a continuum free energy functional at inverse mean-field temperature $\beta_{MF} = \beta N_0$, see also [26]. Letting $N_0 \to \infty$ subsequently, the entropy contribution to the free energy drops out, giving formally Wigner's variational principle. With a leap of faith one may thus expect that the limit $N \to \infty$ in (1.7) will give the same result directly.

Recently, Boutet de Monvel, Pastur, and Shcherbina [5] have studied the limit $N \to \infty$ of the joint eigenvalue distribution of real symmetric random matrix ensembles (1.1) for a large class of $V(M)$, satisfying certain regularity conditions. They prove that in the limit $N \to \infty$ the n-th marginal measure of (1.4), with λ^2 replaced by $V(\lambda)$, factors into an n-fold tensor product of identical one-particle measures whose density is precisely that of the two-dimensional Coulomb fluid restricted to a line, satisfying the equations of electrostatic equilibrium in the applied potential field V. The proof in [5] is based on the classical Stieltjes transform. It is carried out explicitly for $\beta = 1$, but the method covers all $\beta > 0$ (in particular, including $\beta = 2, 4$), as well as certain Hölder continuous many-body interactions.

We here generalize this result in [5] to a wider class of interactions and an arbitrary (finite) number of space dimension. Our result covers also the limit $N \to \infty$ of the eigenvalue distribution of Ginibre's complex and real quaternionic Gaussian random matrix ensembles [20]. However, our method is very different from that in [5]. Instead of using the Stieltjes transform, we adapt the strategy of Messer and Spohn [26], Kiessling [23] and Caglioti et al.[6] to the combined mean-field and zero temperature limit. Interestingly enough, we can work without the detailed control of [5] on the finite-N marginal densities, and in this sense our proof of the variational principle is also considerably shorter and simpler than the proof in [5].

II. MAIN RESULT

We now prepare the statements of our main result. Let $\Lambda \subset \mathbb{R}^{\mathrm{d}}$ be closed and connected, and let dx denote uniform measure on Λ. Note that Λ may be all of $\mathbb{R}^{\mathrm{d}}$, or it may be a lower dimensional manifold, e.g. the sphere $\mathbb{S}^{\mathrm{d}-1}$. We denote by $P(\Lambda)$ the probability measures on Λ, and by $P^s(\Lambda^{\mathbb{N}})$ the permutation symmetric probability measures on the infinite Cartesian product $\Lambda^{\mathbb{N}}$. We recall the decomposition theorem of de Finetti [16] and Dynkin [12] (see also [21,15]), which states that $\mu \in P^s(\Lambda^{\mathbb{N}})$ is uniquely presentable as a linear convex superposition of product measures, i.e., for each $\mu \in P^s(\Lambda^{\mathbb{N}})$ there exists a unique probability measure $\nu(d\varrho|\mu)$ on $P(\Lambda)$, such that for each $n \in \mathbb{N}$,

$$\mu_n(d^n x) = \int_{P(\Lambda)} \nu(d\varrho|\mu)\varrho^{\otimes n}(d^n x)\,, \tag{2.1}$$

where μ_n denotes the n-th marginal measure of μ. This is also the extremal decomposition for the convex set $P^s(\Lambda^{\mathbb{N}})$, see [21].

To establish the limit $N \to \infty$ for measures of the form (1.2), (1.3) only some general properties of the interactions enter. Also, the particular value of β plays no role, and it is convenient to absorb it into the Hamiltonian. Therefore, we study the sequence of probability measures

$$\mu^{(N)}\left(d^N x\right) = \frac{1}{Q^{(N)}} \exp\left(-H^{(N)}(\mathbf{x}_1, ..., \mathbf{x}_N)\right) \prod_{1\leq k\leq N} dx_k \tag{2.2}$$

for Hamiltonians of the form

$$H^{(N)}(\mathbf{x}_1, ..., \mathbf{x}_N) = \sum_{1\leq i<j\leq N} w(\mathbf{x}_i, \mathbf{x}_j) + \sum_{1\leq i\leq N} u(\mathbf{x}_i) + N v(\mathbf{x}_i)\,. \tag{2.3}$$

The pair interaction w and one-particle potentials u, v satisfy the following conditions.

Condition on $w(\mathbf{x}, \mathbf{y})$:

$(C1)$ *Symmetry* : $w(\mathbf{x}, \mathbf{y}) = w(\mathbf{y}, \mathbf{x})$

Condition on $U(\mathbf{x}) = u(\mathbf{x}) + v(\mathbf{x})$:

$(C2)$ *Integrability* : $e^{-U(\mathbf{x})} \in L^1(\Lambda, dx)$

Conditions on $W(\mathbf{x},\mathbf{y}) = w(\mathbf{x},\mathbf{y}) + v(\mathbf{x}) + v(\mathbf{y})$:

$(C3)$ *Lower semicontinuity* : W is l.s.c. on $\Lambda \times \Lambda$

$(C4)$ *Integrability* : $W(\mathbf{x},\mathbf{y}) \in L^1\left(\Lambda^2,\, e^{-U(\mathbf{x})}dx \otimes e^{-U(\mathbf{y})}dy\right)$

In case of an unbounded Λ we also need a growth condition at infinity. Let $W_-(\mathbf{x}) \equiv \min_{\mathbf{y}} W(\mathbf{x},\mathbf{y})$.

$(C5)$ *Confinement* : $\lim_{|\mathbf{x}|\to\infty} W_-(\mathbf{x}) = \infty$, uniformly in $\mathbf{x}$

THEOREM: *Let $\Lambda \subset \mathbb{R}^{\mathrm{d}}$ be closed and connected, and let w, v and u satisfy the conditions $(C1)-(C5)$. Consider (2.2) as extended to a probability on $\Lambda^{\mathbb{N}}$. Then there exists a $\mu \in P^s(\Lambda^{\mathbb{N}})$ such that, after extraction of a subsequence $\mu^{(N')}$,*

$$\lim_{N'\to\infty} \mu^{(N')} = \mu\,. \tag{2.4}$$

For each limit point μ, the decomposition measure $\nu(d\varrho|\mu)$ is concentrated on the subset of $P(\Lambda)$ which consists of the probability measures ϱ that minimize the functional

$$\mathcal{E}(\varrho) = \frac{1}{2}\varrho^{\otimes 2}(W) \tag{2.5}$$

over $P(\Lambda)$.

In general we have little information on the decomposition measure $\nu(d\varrho|\mu)$. More is known for regular mean-field Hamiltonians, see [25]. However, if it can be shown, as is the case for many random matrix ensembles, that (2.5) has a unique minimizer, say ϱ_0, then in (2.4) we have in fact convergence, and the limit is of the form

$$\mu_n = \varrho_0^{\otimes n}. \tag{2.6}$$

As discussed in [35], the factorization property (2.6) is equivalent to a law of large numbers. Consider the eigenvalues averaged over some continuous test function f,

$$\langle f\rangle_N \equiv \frac{1}{N}\sum_{j=1}^{N} f(\lambda_j)\,. \tag{2.7}$$

Then (2.6) implies that, for all such f,

$$\lim_{N\to\infty} \langle f \rangle_N = \int_\Lambda \varrho_0(d\lambda) f(\lambda) \tag{2.8}$$

in probability. We summarize as

COROLLARY: *If (2.5) has a unique minimizer, ϱ_0, then the weak law of large numbers (2.8) holds for all $f \in C^0(\Lambda)$.*

Applications of our theorem to random matrix ensembles are presented in the concluding section.

III. PROOF OF THE THEOREM

The proof looks somewhat technical because we work under fairly minimal assumptions on W. But, in essence, we only have to show, through sharp upper and lower bounds, that the pair-specific free energy converges to the minimum continuum energy. The remainder of the theorem follows from the permutation invariance of the measures.

We define the absolutely continuous (w.r.t. dx) a-priori probability measure on Λ,

$$\mu_0(dx) = Z_0^{-1} e^{-U(\mathbf{x})} dx \,. \tag{3.1}$$

For each $\varrho^{(N)} \in P(\Lambda^N)$ its entropy w.r.t. $\mu_0^{\otimes N}$ is defined by

$$\mathcal{S}^{(N)}\left(\varrho^{(N)}\right) = -\int_{\Lambda^N} \ln\left(\frac{d\varrho^{(N)}}{d\mu_0^{\otimes N}}\right) \varrho^{(N)}(d^N x) \tag{3.2}$$

if $\varrho^{(N)}$ is absolutely continuous w.r.t. a-priori measure $\mu_0^{\otimes N}$, and provided the integral in (3.2) exists. In all other cases, $\mathcal{S}^{(N)}\left(\varrho^{(N)}\right) = -\infty$.

LEMMA 1: *The relative entropy (3.2) is non-positive,*

$$\mathcal{S}^{(N)}\left(\varrho^{(N)}\right) \leq 0 \,. \tag{3.3}$$

Proof of Lemma 1:

A standard convexity argument, see [15]. □

We introduce the symmetric Hamiltonian

$$K^{(N)}(\mathbf{x}_1, ..., \mathbf{x}_N) = \frac{1}{2} \sum_{1 \le j \ne k \le N} W(\mathbf{x}_j, \mathbf{x}_k) \,. \tag{3.4}$$

For $\varrho^{(N)} \in P(\Lambda^N)$, $(\beta\times)$ its Helmholtz free energy (or just free energy) is now defined by

$$\mathcal{F}^{(N)}\big(\varrho^{(N)}\big) = \varrho^{(N)}\big(K^{(N)}\big) - \mathcal{S}^{(N)}\big(\varrho^{(N)}\big) \tag{3.5}$$

if the right side exists, and by $\mathcal{F}^{(N)}\left(\varrho^{(N)}\right) = +\infty$ in all other cases.

LEMMA 2: *The free energy (3.5) takes its unique minimum at the probability measure*

$$\mu^{(N)}(d^N x) = \frac{1}{Z^{(N)}} \exp\Big(-K^{(N)}(\mathbf{x}_1, ..., \mathbf{x}_N)\Big)\, \mu_0^{\otimes N}(d^N x) \,, \tag{3.6}$$

where $Z^{(N)} = Q^{(N)}/Z_0^N$. *Thus,*

$$\min_{\varrho^{(N)} \in P(\Lambda^N)} \mathcal{F}^{(N)}\big(\varrho^{(N)}\big) = \mathcal{F}^{(N)}\big(\mu^{(N)}\big) = -\ln Z^{(N)} \,. \tag{3.7}$$

Proof of Lemma 2:

The variational principle is verified by a standard convexity estimate [15,32], which shows that $\mathcal{F}^{(N)}\big(\varrho^{(N)}\big) - \mathcal{F}^{(N)}\big(\mu^{(N)}\big) \ge 0$, with equality holding if and only if $\varrho^{(N)} = \mu^{(N)}$. The second identity in (3.7) is verified by explicit calculation. □

Notice that the canonical probability measure (3.6) is just (2.2) rewritten in terms of $K^{(N)}$ and μ_0.

LEMMA 3: *The functional* $\mathcal{E}(\varrho)$ *has a finite minimum over* $P(\Lambda)$.

Proof of Lemma 3:

For bounded Λ the lower semicontinuity $(C3)$ establishes that $\mathcal{E}(\varrho)$ is a lower semicontinuous functional for the topology of measures in $P(\Lambda)$. For an unbounded region, the same conclusion holds because of $(C3)$ and $(C5)$. Now the claim follows from standard facts about lower semicontinuous functionals [32]. □

In the following, let ϱ_0 denote a minimizing probability measure for (2.5), and let $E_0 = \mathcal{E}(\varrho_0)$.

LEMMA 4: *The pair specific free energy is bounded above by*

$$\limsup_{N\to\infty}\left(-N^{-2}\ln Z^{(N)}\right)\leq E_0\,. \tag{3.8}$$

We will prove Lemma 4 in two blocks, distinguishing minimizers ϱ_0 with finite entropy from those with negative infinite entropy. For instance, Wigner's semicircle density (1.6) has finite entropy, but other ensembles can have a singular measure as eigenvalue "density."

Proof of Lemma 4

Assume first that the entropy of ϱ_0 is finite, i.e. (recalling Lemma 1),

$$\mathcal{S}^{(1)}\left(\varrho_0\right)=S_0\in(-\infty\,,0]\,. \tag{3.9}$$

Notice that the non-positive constant S_0 is N-independent. We then can use Lemma 2 to estimate for all N that

$$-N^{-2}\ln Z^{(N)}\leq N^{-2}\mathcal{F}^{(N)}(\varrho_0^{\otimes N})=\left(1-N^{-1}\right)\mathcal{E}(\varrho_0)-N^{-1}\mathcal{S}^{(1)}\left(\varrho_0\right)\,. \tag{3.10}$$

The identity in (3.10) follows from $(C1)$. By (3.9), $\mathcal{S}^{(1)}\left(\varrho_0\right)=S_0$, and S_0 is finite and N-independent, and by Lemma 3, $\mathcal{E}(\varrho_0)=E_0$ is finite and N-independent. Thus (3.8) in the finite entropy case follows by taking $N\to\infty$ in (3.10).

Assume now that (3.9) is false, so that $\mathcal{S}^{(1)}(\varrho_0)=-\infty$. In that case, (3.10) becomes useless. Now, since the C_0^∞ functions are dense in $P(\Lambda)$, the obvious way out is to modify the above argument and to work with a regular approximation to ϱ_0. However, since $\mathcal{E}(\varrho)$ is only lower semicontinuous, we have to employ also a continuous approximation to $\mathcal{E}(\varrho)$.

By $(C3)$ for bounded Λ, and by $(C3)$, $(C5)$ in case of an unbounded Λ, W is the pointwise upper limit of a continuous increasing map $\gamma\mapsto W_\gamma\in C^0(\Lambda\times\Lambda)$ that is uniformly bounded below, see [32]. By $(C1)$, we can assume that $W_\gamma(\mathbf{x},\mathbf{y})=W_\gamma(\mathbf{y},\mathbf{x})$. In the following, let $K_\gamma^{(N)}$ be defined by (3.4) with W replaced by W_γ, and let $\mu_\gamma^{(N)}\left(d^Nx\right)$ and $Z_\gamma^{(N)}$, respectively $\mathcal{F}_\gamma^{(N)}\left(\varrho^{(N)}\right)$, be defined by (3.6), respectively (3.5), with $K^{(N)}$ replaced by $K_\gamma^{(N)}$.

Since W satisfies $(C3)$, and W_γ is of class C^0, and $W_\gamma \nearrow W$ pointwise, for each positive $\epsilon \ll 1$ we can find a γ_ϵ such that, simultaneously,

$$E_0 - \frac{1}{2}\varrho_0^{\otimes 2}(W_\gamma) < \frac{1}{3}\epsilon \tag{3.11}$$

and

$$\limsup_{N\to\infty}\left(-N^{-2}\ln Z^{(N)}\right) \leq \limsup_{N\to\infty}\left(-N^{-2}\ln Z_\gamma^{(N)}\right) + \frac{1}{3}\epsilon \tag{3.12}$$

whenever $\gamma \geq \gamma_\epsilon$. Moreover, since W_γ is bounded below on $\Lambda \times \Lambda$ uniformly in γ, and $W_\gamma \in C^0(\Lambda \times \Lambda)$, for each γ_ϵ we can find a measure $\varrho_\epsilon \in P(\Lambda)$ that is equivalent to a positive function of class $C_0^\infty(\Lambda)$, such that

$$\left|\frac{1}{2}\varrho_\epsilon^{\otimes 2}(W_{\gamma_\epsilon}) - \frac{1}{2}\varrho_0^{\otimes 2}(W_{\gamma_\epsilon})\right| < \frac{1}{3}\epsilon\,. \tag{3.13}$$

On the other hand, given any γ and any $\varrho_\delta \in P(\Lambda)$ of class C_0^∞, we have the estimate

$$\begin{aligned} -\ln Z_\gamma^{(N)} &= \mathcal{F}_\gamma^{(N)}\big(\mu_\gamma^{(N)}\big) = \min_{\varrho^{(N)}\in P(\Lambda^N)} \mathcal{F}_\gamma^{(N)}\big(\varrho^{(N)}\big) \\ &\leq \mathcal{F}_\gamma^{(N)}\left(\varrho_\delta^{\otimes N}\right) = N(N-1)\frac{1}{2}\varrho_\delta^{\otimes 2}(W_\gamma) + N\mathcal{S}^{(1)}(\varrho_\delta)\,, \end{aligned} \tag{3.14}$$

where the first line is the analog of Lemma 2, the inequality obvious, and the last identity an explicit computation, using the symmetry of the W_γ. In particular, for any given ϵ we can choose $\gamma = \gamma_\epsilon$ and $\varrho_\delta = \varrho_\epsilon$ in (3.14), then multiply (3.14) by N^{-2} and take the limsup. Since ϱ_ϵ is of class C_0^∞, we have $|\mathcal{S}^{(1)}(\varrho_\epsilon)| = C(\epsilon) < \infty$, independent of N, whence $N^{-1}\mathcal{S}^{(1)}(\varrho_\epsilon) \to 0$. Next we use (3.11), (3.12), (3.13), and the triangle inequality, and conclude

$$\limsup_{N\to\infty}\left(-N^{-2}\ln Z^{(N)}\right) \leq \frac{1}{2}\varrho_\epsilon^{\otimes 2}(W_{\gamma_\epsilon}) + \frac{1}{3}\epsilon \leq E_0 + \epsilon\,, \tag{3.15}$$

for arbitrarily small ϵ. This proves (3.8) for the infinite entropy case.

The proof of Lemma 4 is complete. $\square$

LEMMA 5: *The sequence $N \mapsto \mu^{(N)}$ given by (3.6) is compact for bounded Λ, and tight for unbounded Λ.*

Proof of Lemma 5:

If Λ is bounded then it is also compact, for $\Lambda \subset \mathbb{R}^{\mathrm{d}}$ is closed, and in that case $P(\Lambda)$ is compact for the topology of measures. By Tychonov's theorem, $P^s(\Lambda^{\mathbb{N}})$ is now compact in the product topology. Hence, for bounded Λ, the sequence (3.6) is compact.

If Λ is unbounded, we have to estimate the contribution from outside of B_R^n to the mass of the n-th marginal $\mu_n^{(N)}$ of (3.6), when $N \to \infty$. Recall that a sequence of probability measures $\mu_n^{(N)}$ is tight if for each $\epsilon \ll 1$ there exists a $R = R(\epsilon)$ such that $\mu_n^{(N)}(\mathrm{B}_R^n) > 1-\epsilon$, independent of N, see [11]. Since our marginal measures are compatible (i.e., $\mu_n^{(N)}(d^n x) = \mu_m^{(N)}(d^n x \otimes \Lambda^{m-n})$ for $m > n$) and permutation symmetric by $(C1)$, it suffices to prove tightness for any particular n. We pick $n = 2$.

We notice that by $(C3)$ and $(C5)$,

$$\min_{(\mathbf{x},\mathbf{y})\in\Lambda\times\Lambda} W(\mathbf{x},\mathbf{y}) = W_0 > -\infty\,. \tag{3.16}$$

Since (3.6) is invariant under the transformation $W \to W + C$, we can even assume, without loss of generality, that $W_0 > 0$. We then have the following sandwich bounds, independent of N,

$$0 < \mu_2^{(N)}(W) \leq \mu_0^{\otimes 2}(W)\,, \tag{3.17}$$

with $\mu_0^{\otimes 2}(W) < \infty$, by $(C4)$. The lower bound in (3.17) is obvious, for $W_0 > 0$. To prove the upper bound in (3.17) we use the strategy of [23]. We can replace $K^{(N)}$ by $\alpha K^{(N)}$ in (3.6), with $0 \leq \alpha < \infty$, so that $-2N^{-2}(1 - N^{-1}) \ln Z^{(N)} = \Gamma_N(\alpha)$ is now a function of α. Clearly, $\Gamma_N(0) = 0$, and $W \geq 0$ implies $\Gamma_N(\alpha) \geq 0$ as well as $\Gamma_N'(\alpha) \geq 0$, while the Cauchy-Schwarz inequality implies $\Gamma_N''(\alpha) \leq 0$. Moreover, Jensen's inequality, applied w.r.t. $\mu_0^{\otimes N}$, and $(C4)$ imply

$$\Gamma_N(\alpha) \leq \alpha\mu_0^{\otimes 2}(W)\,. \tag{3.18}$$

Obviously $\mu_0^{\otimes 2}(W)$ is N-independent. Thus, $\Gamma_N(\alpha)$ is a nonnegative, increasing, concave real function, bounded above by (3.18), and mapping zero into itself. A simple geometrical argument now reveals that the slope of any tangent to $\Gamma_N(\alpha)$ never exceeds the slope of the ray on the r.h.s. of (3.18), i.e., $\Gamma_N'(\alpha) \leq \mu_0^{\otimes 2}(W)$. But $\Gamma_N'(1) = \mu_2^{(N)}(W)$, which proves the right inequality in (3.17).

Now pick $\epsilon \ll 1$ arbitrary. By $(C5)$ we can find a $R = R(\epsilon)$ such that

$$\inf_{(\mathbf{x},\mathbf{y}) \notin \mathrm{B}_R^2} W(\mathbf{x},\mathbf{y}) \geq \frac{1}{\epsilon} \mu_0^{\otimes 2}(W) \,. \tag{3.19}$$

Let χ denote the characteristic function of the complement of B_R in Λ. We then have the chain of estimates

$$\begin{aligned} \mu_0^{\otimes 2}(W) &\geq \mu_2^{(N)}(W) \geq \mu_2^{(N)}\big(W\chi^{\otimes 2}\big) \\ &\geq \inf_{(\mathbf{x},\mathbf{y}) \notin \mathrm{B}_R^2} W(\mathbf{x},\mathbf{y})\, \mu_2^{(N)}\big(\chi^{\otimes 2}\big) \geq \frac{1}{\epsilon}\mu_0^{\otimes 2}(W) \left(1 - \mu_2^{(N)}\big(\mathrm{B}_R^2\big)\right) . \end{aligned} \tag{3.20}$$

Division of (3.20) by $\epsilon^{-1}\mu_0^{\otimes 2}(W)$ and a simple rewriting reveals that, independent of N,

$$\mu_2^{(N)}\big(\mathrm{B}_R^2\big) \geq 1 - \epsilon \,, \tag{3.21}$$

which was to be shown. The proof is complete. $\square$

LEMMA 6: *The pair specific free energy is bounded below by*

$$\liminf_{N\to\infty} \left(-N^{-2} \ln Z^{(N)}\right) \geq E_0 \,. \tag{3.22}$$

Proof of Lemma 6:

By Lemma 1, $\mathcal{S}^{(N)}(\mu^{(N)}) \leq 0$. Therefore,

$$-\ln Z^{(N)} \geq \mu^{(N)}\big(K^{(N)}\big) \,. \tag{3.23}$$

By $(C1)$,

$$\frac{1}{N^2}\mu^{(N)}\big(K^{(N)}\big) = \left(1 - \frac{1}{N}\right) \frac{1}{2}\mu_2^{(N)}(W) \,. \tag{3.24}$$

Now pick a converging subsequence of (3.6), $\mu^{(N')} \rightharpoonup \mu \in P^s(\Lambda^{\mathbb{N}})$. Such a converging subsequence exists by Lemma 5 and the Bolzano-Weierstrass theorem. Then, by $(C3)$, we have

$$\liminf_{N'\to\infty} \mu_2^{(N')}(W) \geq \mu_2(W) \,, \tag{3.25}$$

while $1 - {N'}^{-1} \to 1$ trivially. Thus,

$$\liminf_{N\to\infty} \left(-N^{-2} \ln\big(Z^{(N)}\big)\right) \geq \frac{1}{2}\mu_2(W) \,. \tag{3.26}$$

Finally, using the representation (2.1), we see that

$$\frac{1}{2}\mu_2(W) = \int_{P(\Lambda)} \nu(d\varrho|\mu)\,\mathcal{E}(\varrho) \geq \mathcal{E}(\varrho_0)\,, \tag{3.27}$$

and the proof of Lemma 6 is complete. □

Proof of the Theorem.

By Lemma 4 and Lemma 6,

$$\lim_{N\to\infty}\left(-N^{-2}\ln Z^{(N)}\right) = E_0\,. \tag{3.28}$$

Recalling (3.26) and (3.27), we see that (3.28) implies

$$\int_{P(\Lambda)} \nu(d\varrho|\mu)\,\mathcal{E}(\varrho) = \mathcal{E}(\varrho_0) \tag{3.29}$$

for every limit point μ of $\mu^{(N)}$. Equation (3.29) in turn implies that the decomposition measure $\nu(d\varrho|\mu)$ is concentrated on the minimizers of $\mathcal{E}(\varrho)$, for assume not, then

$$\int_{P(\Lambda)} \nu(d\varrho|\mu)\,\mathcal{E}(\varrho) > \mathcal{E}(\varrho_0)\,,$$

which contradicts (3.29). The proof of the Theorem is complete. □

We are now also in the position to vindicate our remark on the existence of the limit in (2.4) in case the minimizer ϱ_0 is unique. Indeed, in that case the set of limit points of $\{\mu^{(N)}, N = 1, 2, \ldots\}$ consists of the single measure.

IV. APPLICATIONS

With the specifications in (2.2), (2.3) that $\mathbf{x} = \lambda \in \Lambda \subset \mathbb{R}^2$, $v(\lambda) = \beta V(\lambda)$, $u(\lambda) = \beta F(\lambda)$, and $w(\lambda,\eta) = \beta G(\lambda,\eta)$, where G is a Green's function for $-\Delta$ in 2D and F its regular part, our theorem characterizes the limit $N = \infty$ of (1.2), (1.3), which for $\beta = 1, 2, 4$ is the joint eigenvalue distribution of various random matrix ensembles of the form (1.1). The decomposition measure of the limit is concentrated on the ground state(s) of the electrostatic energy functional $\varepsilon(\varrho) = \beta^{-1}\mathcal{E}(\varrho)$ of a charged continuum fluid with "charge density" $d\varrho/d\lambda$ (which may be a singular measure) of total charge 1, subject to an external potential V. Explicitly, the energy functional reads

$$\varepsilon(\varrho) = \frac{1}{2}\varrho^{\otimes 2}(G) + \varrho(V)\,. \tag{4.1}$$

The regular part, F, of G does not contribute to the limit. We list a few examples.

IVa. *Real symmetric, complex Hermitian, and quaternionic self-dual Hermitian matrices*

As mentioned in the introduction we have $\Lambda = \mathbb{R}$, $G(\lambda, \eta) = -\ln|\lambda - \eta|$, $\beta = 1, 2, 4$, and $\kappa = 1, 2, 2$, respectively. Our electrostatic variational principle (VP) for (4.1) then becomes the VP of Boutet de Monvel et al. [5], but here with a slightly wider class of potentials V. In particular, for $\beta = 1$ we can allow continuous V with $V(\lambda) \sim (1+\epsilon)\ln|\lambda|$ asymptotically, as compared to Hölder continuous V with $V(\lambda) \sim (2+\epsilon)\ln|\lambda|$ in [5]. The VP has been studied extensively in [33]. A unique minimizer is known to exist under certain regularity conditions on V. Of course, for $V(\lambda) = |\lambda|^2$, the quadratic potential of the Gaussian ensembles, the minimizer of (4.1) is given by Wigner's semicircle law (1.6).

IVb. *General complex matrices*

We have $\Lambda = \mathbb{R}^2$, $G(\lambda, \eta) = -\ln|\lambda - \eta|$. In this case our variational principle for (4.1) generalizes the VP of [5] to two-dimensional domains. Under mild conditions on V, and in particular for all our examples, it can be shown that the minimizer is unique.

We consider only the Gaussian ensemble with $\kappa V(M) = M^\dagger M$ in (1.1), whence $V(\lambda) = |\lambda|^2/2$ in (1.3), and $\beta = 2$ in (1.2). The minimizer of (4.1) is given by

$$d\varrho_0 = \pi^{-1}\chi_{\mathrm{B}_1}(\lambda)\,d\lambda\,, \tag{4.2}$$

where $\chi_{\mathrm{B}_1}(\lambda)$ is the indicator function of the unit disk B_1 in $\mathbb{R}^2$. This result can also be obtained from Ginibre's exact finite N formula, see [20].

IVc. *Complex normal matrices*

We have $\Lambda = \mathbb{R}^2$, $G(\lambda, \eta) = -\ln|\lambda - \eta|$, and $\beta = 2$. Consider first (1.1) with $\kappa V(M) = \ln(1 + M^\dagger M)^{1+1/N}$. Then in (1.3) we have $V(\lambda) = -\ln[\pi\rho_C(|\lambda|)]^{1/2}$, where $\rho_C(\xi) = \pi^{-1}(1+\xi^2)^{-1}$ is the density of the Cauchy distribution, and F is replaced by V. With these identifications $(C5)$ is violated, but (1.2) is well defined for all $\beta > 1$, and the minimizer of the electrostatic energy functional is found to be

$$d\varrho_0 = \pi^{-1}(1+|\lambda|^2)^{-2}d\lambda\,. \tag{4.3}$$

The measure (4.3) has geometrical significance. Recall that $|J|^2(\lambda) = 4/(1+|\lambda|^2)^2$ is the Jacobian of the stereographic projection map $\mathbb{S}^2 \to \mathbb{R}^2$, arranged such that the

equator of $\mathbb{S}^2$ coincides with the unit circle in $\mathbb{R}^2$. Therefore, (4.3) is the stereographic projection onto the Euclidean plane of the uniform probability measure on $\mathbb{S}^2$. Also the finite-N measure (1.2), with (1.3) specified as above, is itself a stereographic projection onto Euclidean space of a canonical ensemble measure of N point charges in the two-sphere $\mathbb{S}^2$. For $\beta = 2$, this *spherical ensemble* is given by (1.2) with Hamiltonian

$$H^{(N)}(\lambda_1, ..., \lambda_N) = - \sum_{1 \leq j < k \leq N} \ln |\lambda_j - \lambda_k| \,, \tag{4.4}$$

where $\lambda_j \in \mathbb{S}^2$, $|\lambda_j - \lambda_k|$ is the chordal distance on $\mathbb{S}^2$, and $d\lambda_k$ in (1.2) now means uniform measure on $\mathbb{S}^2$. (If $\beta \neq 2$, a non-constant one-particle potential has to be added to $H^{(N)}$). Our theorem applies directly to this ensemble on $(\mathbb{S}^2)^{\times N}$. We arrive at (4.3) by taking the limit $N \to \infty$ on the sphere, arguing that the minimizer is a constant, and projecting the result onto the Euclidean plane.

In a sense, the spherical ensemble is the counterpart on $\mathbb{S}^2$ to Dyson's circular ensembles on $\mathbb{S}^1$, although these ensemble are related to random matrices in a different manner. The statistical mechanics of the spherical ensemble has been studied in some detail in [7,19], using exact algebraic techniques. The spherical ensemble is also related to J.J. Thomson's celebrated problem of determining the minimum energy configuration of N point charges on $\mathbb{S}^2$ [36], which recently has received much attention in physics [3], topology and Knot theory [24].

As a second example, consider entries restricted by $\|M\|_\infty \leq 1$ for every M, with uniform distribution otherwise. This corresponds to $V(M) = \lim_{t\to\infty}(1 + \tanh[t(I - MM^\dagger)])^{-1}$. Alternatively, we can choose $\Lambda = \mathrm{B}_1$ and $V = 0$ in (1.2). The unique minimizer of the corresponding variational principle is the electrostatic charge distribution on a circular perfect conductor. It is well known that any surplus charge accumulates at the "surface," i.e., the minimizer is given by a Dirac mass concentrated uniformly on the boundary of the unit disk,

$$d\varrho_0 = \delta_{\mathbb{S}^1}(dx) \,. \tag{4.5}$$

With the help of our corollary this gives us the following.

Proposition: *Let $\langle f \rangle_N$ be defined as in (2.7), the summation running over the eigenvalues, restricted to $B_1 \subset \mathbb{C}$, of a complex normal $N \times N$ random matrix whose free entries*

are uniformly distributed otherwise. Then, in probability, $\langle f \rangle_N \to (2\pi)^{-1} \int_0^{2\pi} f\left(e^{i\varphi}\right) d\varphi$ *as* $N \to \infty$.

This result is worth rephrasing in less technical terms. As far as averages over the spectrum are concerned, an infinite normal random matrix with eigenvalues in the unit disk and independent entries uniformly distributed otherwise, is almost surely equivalent to some infinite unitary matrix.

IVd. *Real quaternion matrices*

We consider only the Gaussian ensemble. The joint probability density is given by (1.5) with $\Lambda = \mathbb{R}^2$, $\beta = 2$, $\kappa = 1$, [20,28]. The limiting eigenvalue distribution now minimizes the slightly more complicated electrostatic energy functional

$$\varepsilon(\varrho) = \frac{1}{2}\varrho^{\otimes 2}\left(-\ln|\lambda - \eta| - \ln|\lambda - \eta^*| + |\lambda|^2 + |\eta|^2\right) , \tag{4.6}$$

where η^* is the mirror image of η with respect to the real axis. The minimizer of (4.6) is, once again, unique and given by the measure (4.2).

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